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DIFFERENTIABILITY VIA DIRECTIONAL DERIVATIVES

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ABSTRACT. Let F be a continuous function from an open subset D of a separable Banach space X into a Banach space Y. We show that if there is a dense G_δ subset A of D and a G_δ subset H of X whose closure has nonempty interior, such that for each $a \in A$ and each $x \in H$ the directional derivative $D_xF(a)$ of F at a in the direction x exists, then F is Gâteaux differentiable on a dense G_δ subset of D. If X is replaced by R^n, then we need only assume that the n first order partial derivatives exist at each $a \in A$ to conclude that F is Frechet differentiable on a dense, G_δ subset of D.

1. Introduction. It is known that if F is a real valued function on an open subset D in R^n whose first order partial derivatives exist and are continuous, then F is differentiable on D. Stepanoff [7] showed that there exists a continuous function on R^2 such that the partial derivatives exist almost everywhere but the function is nowhere differentiable. In contrast to this result, we prove in this note that the existence of partial derivatives on a dense G_δ subset will imply that the function is differentiable on a dense G_δ subset.

Throughout, we assume the scalar field is R. Let X and Y be locally convex spaces and let F be a function from an open subset D of X into Y. We say that F is *Gâteaux differentiable* at $a \in D$ if there exists a continuous linear operator $DF(a)\colon X \to Y$ such that for any $x \in X$,

$$DF(a)(x) = \lim_{t\to 0} t^{-1}(F(a + tx) - F(a))$$

(the limit depends on x). We call $DF(a)$ the *Gâteaux differential* of F at a. When the above limit exists it is called the directional derivative of F at a in the direction x and is denoted by $D_xF(a)$. The reader may refer to [9] for a detailed discussion of the Gâteaux derivative and its relations with other derivatives. Our main theorem is the following: Suppose F is a continuous function from an open subset D of a separable Banach space X into a Banach space Y and suppose there exists a dense G_δ subset A of D and a G_δ subset H of X whose closure has nonempty interior such that for $a \in A$, the directional derivative $D_xF(a)$ exists for each $x \in H$. Then F is Gâteaux differentiable on a dense G_δ of D.

In §2, we give some lemmas. In §3, we prove the main theorem as well as some other related results.

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2. Some lemmas. A (Hausdorff) topological space X is called a *Baire space* if the intersection of any sequence of open dense subsets in X is dense in X. It may be shown that dense G_δ subsets and open subsets of Baire spaces and, of course, complete metric spaces are Baire spaces.

We omit the proof of the following obvious lemma.

LEMMA 2.1. *Let G_n be a sequence of open subsets in a Baire space X with $\bigcup_{n=1}^{\infty} G_n$ dense in X. Let A be a subset of X such that for each n, $A \cap G_n$ contains a dense G_δ in G_n. Then A contains a dense G_δ in X.*

Let X and Y be two sets and let A be a subset of $X \times Y$. For each $x \in X$, we use A_x to denote the x-section of A; i.e. $A_x = \{y \in Y: (x, y) \in A\}$.

LEMMA 2.2. *Let X and Y be Baire spaces with Y second countable, and let A be a dense G_δ subset of $X \times Y$. Then $\{x \in X: A_x$ is a dense G_δ subset of $Y\}$ contains a dense G_δ subset of X.*

PROOF. Assume first that A is an open dense subset of $X \times Y$. Let $A = \bigcup_\alpha (U_\alpha \times V_\alpha)$, where U_α and V_α are open subsets of X and Y, respectively, let $\{W_n\}$ be a countable base for the topology on Y, and for each n let $B_n = \bigcup_\alpha \{U_\alpha: V_\alpha \cap W_n \neq \emptyset\}$. It is clear that B_n is open and easy to show that B_n is dense in X. Since X is a Baire space, $B = \bigcap_{n=1}^{\infty} B_n$ is a dense G_δ subset of X. Let $x \in B$. For each n there exists an α such that $x \in U_\alpha$ and $V_\alpha \cap W_n \neq \emptyset$. It follows that A_x is dense in Y. Noting that A_x is open completes the proof in the case where A is open.

Let $A = \bigcap_{n=1}^{\infty} A_n$ where each A_n is an open dense subset of $X \times Y$. For each n there is a dense G_δ subset B_n of X such that $B_n \subseteq \{x \in X: (A_n)_x$ is a dense G_δ subset of $Y\}$. Let $B = \bigcap_{n=1}^{\infty} B_n$. Since X is a Baire space, B is a dense G_δ subset of X. Let $x \in B$. Then for each n $(A_n)_x$ is a dense G_δ subset of Y. Since Y is a Baire space, $\bigcap_{n=1}^{\infty} (A_n)_x$ is a dense G_δ subset of Y. But $\bigcap_{n=1}^{\infty} (A_n)_x = A_x$, which completes the proof.

LEMMA 2.3. *Let X be a Baire space, let Y be a metric space, and let F: $X \times \{t: 0 < |t| < r\} \to Y$, where r is a fixed positive number. Suppose that for each $0 < |t| < r$, $F(x, t)$ is continuous in x, and that there is a dense G_δ set A in X such that $\lim_{t \to 0} F(a, t) = F_0(a)$ exists for all $a \in A$. Then there is a dense G_δ set E in X such that for each $a \in E$,*

$$\lim_{(x, t) \to (a, 0),\, t \neq 0} F(x, t) = F_0(a).$$

PROOF. F_0 is a function of Baire class 1. Let $E_1 = \{a \in A: F_0$ is continuous at $a\}$. Then E_1 is a dense G_δ set in A and, hence, in X [3, Theorem 2]. Let M and N be positive integers, and let

$$A_{MN} = \{x \in X: \text{if } 0 < |t| < N^{-1} \text{ and } 0 < |s| < N^{-1}, \text{ then } \rho(F(x, t), F(x, s)) \leqslant M^{-1}\}.$$

Then A_{MN} is a closed set and, for each M, $\bigcup_{N=1}^{\infty} A_{MN} \supseteq A$. So since X is a Baire space, $G_M = \bigcup_{N=1}^{\infty} A_{MN}^0$ is an open dense subset of X. Let $E_2 =$

$\bigcap_{M=1}^{\infty} G_M$. Since X is a Baire space, E_2 is a dense G_δ set in X as is $E = E_1 \cap E_2$.

Let $a \in E$ and let $\varepsilon > 0$. Since F_0 is continuous at a there is a neighborhood U_1 of a such that if $y \in U_1 \cap A$, then $\rho(F_0(y), F_0(a)) < \varepsilon/2$. There is a positive integer M such that $M^{-1} < \varepsilon/2$. Since $a \in E_2$, $a \in G_M$. Thus there is an N such that $a \in A_{MN}^0$. Let $U = U_1 \cap A_{MN}^0$. Let $x \in U$ and $0 < |t| < N^{-1}$. Then for each $y \in U \cap A$,

$$\rho(F(x, t), F_0(a)) \leqslant \rho(F(x, t), F(y, t)) + \rho(F(y, t), F_0(y)) + \rho(F_0(y), F_0(a)).$$

Since $U \cap A$ is dense in U and since $F(y, t)$ is continuous as a function of y,

$$\lim_{y \to x} \rho(F(x, t), F(y, t)) = 0.$$

Since $y \in U \cap A$, and since $0 < |t| < N^{-1}$,

$$\rho(F(y, t), F_0(y)) = \lim_{s \to 0} \rho(F(y, t), F(y, s)) \leqslant M^{-1}.$$

Thus

$$\limsup_{y \to x} \rho(F(y, t), F_0(y)) \leqslant M^{-1} < \varepsilon/2.$$

Consequently, $\rho(F(x, t), F_0(a)) < \varepsilon$.

LEMMA 2.4. *Let X be a Banach space and let A be a dense G_δ in an open subset U of X. Then the linear hull of A equals X.*

PROOF. Suppose $\operatorname{lin} A \neq X$. Then there exists an x in $X \setminus \operatorname{lin} A$ so that $U \cap (U + x) \neq \varnothing$. It is easy to see that both $A \cap U \cap (U + x)$ and $(A + x) \cap U \cap (U + x)$ contain dense G_δ subsets of $U \cap (U + x)$ and are disjoint, which is a contradiction.

3. The theorems.

THEOREM 3.1. *Let X and Y be separable Banach spaces. Let D be an open subset of X, and let F: $D \to Y$ be continuous. Suppose there is a dense G_δ subset A of D such that for each $a \in A$, the directional derivative $D_x F(a)$ exists for each $x \in H$ where H is a G_δ subset of X which is dense in a nonempty open subset U of X. Then F is Gâteaux differentiable on a dense G_δ subset of D.*

PROOF. For each positive integer k let

$$A_k = \left\{(a, x) \in D \times U : \left\|t^{-1}(F(a + tx) - F(a))\right\| \leqslant k \text{ for } 0 < |t| < k^{-1}\right\}.$$

Since F is continuous, A_k is closed, and since $D_x F(a)$ exists for each $a \in A$ and $x \in H$, $A \times H \subseteq \bigcup_{k=1}^{\infty} A_k$. Thus $\bigcup_{k=1}^{\infty} A_k$ is dense in $D \times U$, which is an open subset of the complete metric space $X \times X$ and, consequently, a Baire space. It follows that $\bigcup_{k=1}^{\infty} A_k^0$ is dense in $D \times U$. Note that each A_k^0 is the countable union of $D_{kj} \times U_{kj}$, where D_{kj} is open in D and U_{kj} is open in U. By Lemma 2.1 it suffices to prove that for each j D_{kj} contains a suitable dense G_δ set. For notational ease D_{kj} and U_{kj} will be denoted simply by D_k and U_k, respectively.

Define $\tilde{F}$: $D_k \times U_k \times \{t: 0 < |t| < k^{-1}\} \to Y$ by

$$\tilde{F}(a, x, t) = t^{-1}(F(a + tx) - F(a)).$$

For each $(a, x) \in (A \cap D_k) \times (H \cap U_k)$, $\lim_{t\to 0} \tilde{F}(a, x, t) = D_xF(a)$. $A \cap D_k$ and $H \cap U_k$ are dense G_δ subsets of D_k and U_k, respectively. Hence $(A \cap D_k) \times (H \cap U_k)$ is a dense G_δ subset of $D_k \times U_k$ and, as was pointed out in the above paragraph, $D_k \times U_k$ is a Baire space. So by Lemma 2.3 there exists a dense G_δ subset E of $(A \cap D_k) \times (H \cap U_k)$ such that for each $(a, x) \in E$,

$$\lim_{(a', x', t)\to(a, x, 0),\, t\neq 0} \tilde{F}(a', x', t) = D_xF(a).$$

By Lemma 2.2 there is a dense G_δ subset B of $A \cap D_k$, which is therefore a dense G_δ subset of D_k, such that for each $a \in B$, E_a is a dense G_δ subset of $H \cap U_k$ and, consequently, a dense G_δ subset of U_k.

Let $a \in B$, $x \in E_a$ and $y \in X$ such that $D_yF(a)$ exists. Then

$$\begin{aligned}
&\lim_{t\to 0} t^{-1}\big(F(a + t(x + y)) - F(a)\big)\\
&\qquad = \lim_{t\to 0} t^{-1}(F(a + tx + ty) - F(a + ty))\\
&\qquad\quad + \lim_{t\to 0} t^{-1}(F(a + ty) - F(a))\\
&\qquad = \lim_{(a+ty, x, t)\to(a, x, 0)} \tilde{F}(a + ty, x, t) + D_yF(a)\\
&\qquad = D_xF(a) + D_yF(a).
\end{aligned}$$

So $D_{x+y}F(a)$ exists and $D_{x+y}F(a) = D_xF(a) + D_yF(a)$. Using the obvious fact that if $D_xF(a)$ exists and if $\alpha \in R$, then $D_{\alpha x}F(a)$ exists and $D_{\alpha x}F(a) = \alpha D_xF(a)$, it is an easy matter to establish that

$$\{x \in X: D_xF(a) \text{ exists and for each } y \text{ such that } D_yF(a) \text{ exists, } D_{x+y}F(a) \text{ exists and } D_{x+y}F(a) = D_xF(a) + D_yF(a)\}$$

is a subspace of X. Since it contains E_a Lemma 2.4 implies that it is X. That is, for all $x \in X$, $D_xF(a)$ exists and for each $y \in X$, $D_{x+y}F(a) = D_xF(a) + D_yF(a)$. Thus the operator $(DF(a))(x) = D_xF(a)$ is linear. Since $D_k \times U_k \subseteq A_k$, for all $x \in U_k$, $\|D_xF(a)\| \leqslant k$; that is, the linear operator is bounded on a nonempty open set U_k and, hence, is bounded.

If $X = R^n$, we let $\{e_1, \ldots, e_n\}$ be the natural basis, and let D_iF denote the partial derivative of F in the ith coordinate.

THEOREM 3.2. *Let $D \subseteq R^n$ be open, let Y be a separable Banach space, and let F: $D \to Y$ be continuous. Suppose there is a dense G_δ subset A of D such that for each $a \in A$ and each $i = 1, \ldots, n$, $D_iF(a)$ exists. Then F is (Frechet) differentiable on a dense G_δ subset of D.*

PROOF. For $r > 0$ let $D_r = \{x \in D: \operatorname{dist}(x, R^n \setminus D) > r\}$. By Lemma 2.1 we need only prove the theorem on D_r for each $r > 0$. For each $i = 1, \ldots, n$ define $\tilde{F}_i: D_r \times \{t: 0 < |t| < r\}$ by

$$\tilde{F}_i(a, t) = t^{-1}(F(a + te_i) - F(a)).$$

Then for each $i = 1, \ldots, n$ and each $a \in A \cap D_r$, $\lim_{t\to 0} \tilde{F}_i(a, t) = D_iF(a)$.

By Lemma 2.3 for each $i = 1, \ldots, n$ there is a dense G_δ subset A_i of D_r such that for each $a \in A_i$,

$$\lim_{(a', t)\to(a, 0),\, t\neq 0} \tilde{F}(a', t) = D_iF(a).$$

Let $E = \bigcap_{i=2}^{n} A_i$. Then E is a dense G_δ subset of D_r.

Let $a \in E$. For each $x \in R^n \setminus \{0\}$ write $x = \sum_{j=1}^{n} \lambda_j e_j$, and for each $i = 2, \ldots, n + 1$, let $a_i = a + \sum_{j=1}^{i-1} \lambda_j e_j$. Then

$$\begin{aligned}
\lim_{x\to 0} |x|^{-1} &\left|\left(F(a + x) - F(a) - \sum_{i=1}^{n} \lambda_i D_iF(a)\right)\right| \\
&\leqslant \sum_{i=2}^{n} \lim_{x\to 0} |x|^{-1}|(F(a_{i+1}) - F(a_i) - \lambda_i D_iF(a))| \\
&\quad + \lim_{x\to 0} |x|^{-1}|(F(a + \lambda_1 e_1) - F(a) - \lambda_1 D_1F(a))| \\
&\leqslant \sum_{i=2}^{n} \lim_{(a_i, \lambda_i)\to(a, 0)} |\tilde{F}(a_i, \lambda_i) - D_iF(a)| \\
&\quad + \lim_{\lambda_1\to 0} |\lambda_1^{-1}(F(a + \lambda_1 e_1) - F(a)) - D_1F(a)| \\
&= 0.
\end{aligned}$$

Suppose F is a continuous function from an open subset D of R^n into a Banach space Y and suppose that D_iF, $i = 1, \ldots, n$, exists on a dense G_δ subset A in D. Then F satisfies the local Lipschitz condition on an open, dense subset in D. Indeed, if for each positive integer k,

$$A_k = \left\{a \in D: \|t^{-1}(F(a + te_i) - F(a))\| \leqslant k, \quad |t| \leqslant k^{-1}, i = 1, \ldots, n\right\},$$

then each A_k is closed and $\bigcup_{k=1}^{\infty} A_k$ contains the set A. Hence the set $U = \bigcup_{k=1}^{\infty} A_k^0$ is an open dense set in D and F satisfies the local Lipschitz condition at each point $a \in U$.

THEOREM 3.3. *Let F be a continuous function from an open subset $D \subseteq R^n$ into R^m. Suppose there exists a dense G_δ subset $A \subseteq D$ such that $D_iF(a)$, $i = 1, \ldots, n$, exist for each $a \in D$. Then F is differentiable on a dense, measurable subset in D with positive (Lebesgue) measure.*

PROOF. It follows from the above remark that F satisfies the local Lipschitz condition on an open, dense subset U in D. By a theorem of Rademacher ([6], cf. also [2, p. 218]), F is differentiable a.e. on U and, hence, F is differentiable on a dense set with positive measure.

The theorem of Rademacher on the differentiability of Lipschitz mappings has been generalized by Mankiewicz [4], [5] and Christensen [1] into some classes of infinite dimensional Banach spaces. However, we remark that a Lipschitz function is not necessarily differentiable on any dense G_δ subset; an example was given by Zahorski in [8].

In the rest of this section, we will consider a map F from an open subset D of a separable Banach space X into a dual Banach space Y^* with the weak* topology. The definition of Gâteaux differentiable of F at a becomes: there exists a bounded linear operator $DF(a)$ from X into Y^* such that

$$\langle DF(a)x, y\rangle = \lim_{t\to 0} t^{-1}\langle F(a+tx) - F(a), y\rangle \text{ for all } x \in X, y \in Y.$$

For convenience, we will call this the *w^*-Gâteaux differential* of F at a. We also call a function $F\colon D \to Y^*$ *demicontinuous* if F is continuous with respect to the weak* topology in Y^*.

THEOREM 3.4. *Let X and Y be separable Banach spaces. Let D be an open subset of X and let F be a demicontinuous function from D into Y^*. Suppose there exists a dense G_δ subset A of D such that for each $a \in A$, $D_xF(a)$ exists for each direction $x \in X$. Then F is w^*-Gateaux differentiable on a dense G_δ in D.*

PROOF. For each $a \in A$, $x \in X$, $0 < \delta$ sufficiently small and $y \in Y$, $t^{-1}\langle F(a+tx) - F(a), y\rangle$ is a continuous function for $|t| \leqslant \delta$. Consequently, $\{t^{-1}\langle F(a+tx) - F(a), y\rangle\colon 0 < |t| < \delta\}$ is a bounded set. By the uniformly boundedness principle, the set $\{\|t^{-1}(F(a+tx) - F(a))\|\colon 0 < |t| < \delta\}$ is also a bounded set. Let

$$A_k = \left\{(a, x) \in D \times X\colon \left\|t^{-1}\left(F(a+tx) - F(a)\right)\right\| \leqslant k \text{ for all } |t| < k^{-1}\right\}.$$

By the demicontinuity and the lower semicontinuity of the norm, each A_k is closed in $D \times X$. Since $\bigcup_{k=1}^{\infty} A_k \supseteq A \times X$, the set $\bigcup_{k=1}^{\infty} A_k^0$ is an open dense subset in $D \times X$.

It follows from the same argument as Theorem 3.1 that it suffices to prove the theorem on any open subset D_k where $D_k \times U_k \subseteq A_k^0$. The rest of the proof is also the same as Theorem 3.1; the only changes are: (i) when applying Lemma 2.3 to $\tilde{F}$, we have to observe that Y is separable, the image of $\tilde{F}$, which is contained in a bounded set in Y^*, is w^*-metrizable; (ii) in the proof of $D_{x+y}F(a) = D_xF(a) + D_yF(a)$ we change

$$\lim_{t\to 0} t^{-1}\big(F(a + t(x+y)) - F(a)\big) = \text{etc.}$$

into

$$\lim_{t\to 0}\left\langle t^{-1}\big(F(a + t(x+y)) - F(a)\big), y\right\rangle = \text{etc.}$$

for each $y \in Y$.

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